Permutation

1. Numbers that can be formed = ${}_4P_3 \times {}_5P_2 = 4 \times 3 \times 2 \times 1 \times 5 \times 4 = \underline{\underline{480}}$

2. If the thousandth digit is 3, 4 or 5, the number of 4 – digit numbers = $3 \times {}_5P_3 = 3 \times 5 \times 4 \times 3 = 180$

If the thousandth digit is 2, the number of 4 – digit numbers greater than 2100 can be formed

$= 3 \times {}_4P_2$ (the hundredth place is 3, 4 or 5) + ${}_4P_2$ (the hundredth place is 1) = 48

∴ Total number of required 4 – digit number = 180 + 48 = $\underline{\underline{228}}$

3. **(i)** The number of 9 – digit numbers that can be formed = 9! = $\underline{\underline{362880}}$

(ii) There are 5 odd numbers and 4 even numbers in 1, 2, …, 9

Therefore there are 4/9 of the numbers in (i) are even.

Of all such even numbers there are 1/2 of them are divisible by 4.

The possible numbers = $9! \times 4/9 \times 1/2 = \underline{\underline{80640}}$

4. Assume that the 3 English books as if it were 1 set.

Together with 5 Chinese books, there are 6 sets.

Total permutation of 6 set = 6!

But the English books may permutate among themselves, number of permutations = 3!

Therefore total number of arrangements = $6! \times 3!$ = $\underline{\underline{4320}}$

5. If repetitions are not allowed, number of 5 – digit numbers formed = 5! = $\underline{\underline{120}}$

Since each digit place from unit place to ten-thousandth place, each number 1, 2, 3, 4, 5 may have the same c

hence to become the place value of the 5–digit number, the mean place value is (1+2+3+4+5)/5 = 3

Therefore the mean of the 120 numbers is 33333.

The sum = 120×33333 = $\underline{\underline{3999960}}$

If repetition is allowed, the number of 5–digit numbers formed = 55 = $\underline{\underline{3125}}$

The required sum = $33333 \times 5^5 = 3 \times 5^5 \times 11111 = \underline{\underline{104165625}}$

6. Total permutation of 8 books = 8!

If two particular must be placed together, the number of arrangements = $2 \times 7!$

Therefore if two books must be separated, the number of arrangements = $8! - 2 \times 7! = \underline{\underline{30240}}$

7. The circular permutation for 5 gentlemen $= (5-1)! = 4!$

After the gentlemen has taken their seats, the ladies must sit between two of the gentlemen,

the permutation of 5 ladies $= 5!$

Therefore the total arrangements $= 4! \times 5! = \underline{\underline{2880}}$

8. **(a)** If the order of a, o, i can be changed the number of ways $= 9! = 962880$

If the order cannot be changed, the number of ways $= 9!/3! = \underline{\underline{60480}}$

(b) The number of ways in which the order of consonants cannot be changed $= 9!/6! = \underline{\underline{504}}$

(c) The number of possible ways $= 9!/(3!6!) = \underline{\underline{84}}$

9. Consider the L.H.S. of the central mark. There must be m white counters and n red counters in order to have symmetry and the total number of counters on L.H.S. $= m + n$.

Total number of arrangements in L.H.S. $= (m + n)!/(m!n!)$

The counters of R.H.S. must be symmetric about the central mark.

Given any arrangement of L.H.S., there is only 1 way of setting the R.H.S.

therefore the total number of ways $= \underline{\underline{\dfrac{(m+n)!}{m!n!}}}$

10. We may consider all the letters x are the same and all the letters y are the same.

After taken such $2n$ x and y letters out to form an arrangement, we then number the x and y letters in ascending orders,

Since there are $2n$ letters, n x-letters and n y-letters, the total number of arrangements $= \underline{\underline{\dfrac{(2n)!}{n!n!}}}$

11. **(i)** Place a nought on the left-most end.

For the rest, either we place a nought or we place a unit consisting of a cross followed by a nought (x, o).

As there are q number of such units, the number of remaining noughts is $(p-q-1)$.

Total number of arrangements $= \dfrac{[(p-q-1)+q]!}{(p-q-1)!q!} = \underline{\underline{\dfrac{(p-1)!}{(p-q-1)!q!}}}$.

(ii) Give each job a nought and each man a cross. The question is reduced to part (i) as in the followings: If a cross is followed by k noughts, it means that the man gets the k jobs. The leftmost end must be a cross. After placing this cross (1 man used), every thing is the same as in (i) and we have the same number of arrangements. i.e. $\underline{\underline{\dfrac{(p-1)!}{(p-q-1)!q!}}}$.

Combination

1. **(a)** Number of straight lines that can be formed from n points $= {}_nC_2$.

Since p points are in a straight line, therefore only 1 line can be formed instead of pC2 lines.

Therefore total number of straight lines that can be formed $= \underline{\underline{{}_nC_2 - {}_pC_2 + 1}}$.

(b) Number of triangles that can be formed from n points $= {}_nC_3$.

Since p points are in a straight, they can't form any triangle.

Therefore ${}_pC_3$ triangles formed from p points must be neglected.

Therefore total number of triangles $= \underline{\underline{{}_nC_3 - {}_pC_3}}$.

2. Number of triangles that can be formed by joining any three vertices of a polygon with n sides $= {}_nC_3$.

By taking any one special vertex on the polygon, the no. of triangles with common sides with the polygon $= n - 2$. However, there are two triangles with two sides in common with the sides of the polygon. These two triangles are duplicated when we consider other vertices. Therefore number of triangles formed from one special vertex without duplication $= n - 2 - 1 = n - 3$.

Total number of triangles with sides common to the polygon $= n(n - 3)$.

$\therefore$ Number of triangles that can be formed with no sides in common with the polygon

$= {}_nC_3 - n(n - 3) = \underline{\underline{\frac{1}{6}n(n-4)(n-5)}}$.

3. By choosing any two lines from the 10 parallel lines $({}_{10}C_2)$ and by choosing any two lines $({}_7C_2)$, we can form a parallelogram, Therefore the total number of parallelograms $= {}_{10}C_2 \times {}_7C_2 = \underline{\underline{135}}$

4. The number of ways of choosing the committee $= {}_8C_3 \times {}_7C_3 = 56 \times 35 = \underline{\underline{1960}}$

When Mr. Y is a member, number of combination $= {}_7C_3 \times {}_6C_3 = 35 \times 20 = 700$

When Miss X is a member, number of combination $= {}_7C_2 \times {}_6C_4 = 21 \times 15 = 315$

When both Mr. Y and Miss X are not members, number of combination $= {}_7C_3 \times {}_6C_4 = 35 \times 15 = 525$

Therefore total number of combinations $= 700 + 315 + 525 = \underline{\underline{1540}}$

5. Number of ways in a committee can be formed $= {}_6C_2 \times {}_7C_3 \times {}_3C_1 = \underline{\underline{945}}$

Number of ways so that a particular Frenchman belong to the committee $= {}_5C_1 \times {}_6C_1 \times {}_3C_1 = \underline{\underline{90}}$

6. For any two circles, there are 2 greatest number of intersection points.

Therefore the greatest number of points of intersection between circles themselves $= 2 \times {}_nC_2 = n(n - 1)$

The greatest number of points of intersection between 1 line and 1 circle is 2 .

Therefore the greatest number of points of intersection between m lines and n circles $= 2mn$

The number of points of intersection between 2 lines is 1 .

Therefore the greatest number of points of intersection between m lines = ${}_mC_2 = m(m-1)/2$

Total number of intersections $= \underline{\underline{\frac{1}{2}m(m-1) + n(2m+n-1)}}$.

Miscellaneous

1. Choose 0 thing from those alike, n things from those different, number of ways $= {}_nC_n$.

Choose 1 thing from those alike, (n – 1) things from those different, number of ways $= {}_nC_{n-1}$.

Choose 2 things from those alike, (n – 2) things from those different, number of ways $= {}_nC_{n-2}$.

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Choose n things from those alike, 0 things from those different, number of ways $= {}_nC_0$.

$\therefore$ Total number of selections $= {}_nC_n + {}_nC_{n-1} + {}_nC_{n-2} + \ldots + {}_nC_0 = \underline{\underline{2^n}}$.

2. Let the two particular objects be A and B.

If A is in the selection, B is not in the selection, the number of ways $= {}_{n-1}C_{r-1}$.

If B is in the selection, A is not in the selection, the number of ways $= {}_{n-1}C_{r-1}$.

If A and B are not in the selection, the number of ways $= {}_{n-2}C_r$.

$\therefore$ Total number of selections $= {}_{n-2}C_{r-1} + {}_{n-2}C_{r-1} + {}_{n-2}C_r = \underline{\underline{\dfrac{(n+r-1)(n-2)!}{r!(n-r-1)!}}}$.

3. First we choose 5 postcards to place in A , the number of ways $= {}_{10}C_5$.

Then we choose from the remaining 5 cards, 3 cards to place in B , the number of ways $= {}_5C_3$.

The rest must be placed in C.

$\therefore$ Total number of selections $= {}_{10}C_5 \times {}_5C_3 = \underline{\underline{2520}}$.

4. The number of ways of labeling $= \dfrac{9!}{4!3!2!} = \underline{\underline{1260}}$

5. $\dfrac{(np)!}{(p!)^n n!}$ is an integer because it is the number of ways of dividing up np different objects into unordered batches of p elements each.

6. (2n + 1)(2n + 3)(2n + 5)…(4n – 3)(4n – 1)

$$= \prod_{i=1}^{n}(2n+2i-1) = \prod_{i=1}^{n}\frac{(2n+2i-1)(2i)}{2i} = \prod_{i=1}^{n}\frac{1}{2}\prod_{i=1}^{n} i \prod_{i=1}^{n}\frac{2(2n+2i-1)}{i} = \left(\frac{1}{2}\right)^n n! \prod_{i=1}^{n}\frac{2i(n+i)^2(2n+2i-1)}{i^2(n+i)^2}$$

$$= \left(\frac{1}{2}\right)^n n! \frac{\prod_{i=1}^{n} i(n+i)(2n+2i)(2n+2i-1)}{\prod_{i=1}^{n} i(n+i)\prod_{i=1}^{n} i(n+i)} = \underline{\underline{\left(\frac{1}{2}\right)^n \frac{n!(4n)!}{(2n)!(2n!)}}} .$$